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Changes of the mean velocity profiles in the combined wave–current motion described in a GLM formulation

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The generalized Lagrangian mean (GLM) formulation is used to describe the interaction of waves and currents. In contrast to the more conventional Eulerian formulation the GLM description enables splitting of the mean and oscillating motion over the whole depth in an unambiguous and unique way, also in the region between wave crest and trough. The present paper deals with non-breaking long-crested regular waves on a current using the GLM formulation coupled with a WKBJ-type perturbation-series approach. The waves propagate under an arbitrary angle with the current direction. The primary interest concerns nonlinear changes in the vertical distribution of the mean velocity due to the presence of the waves, but modifications of the orbital velocity profiles, due to the presence of a current, are considered as well. The special case of no initial current, where waves induce a so-called drift velocity or mass-transport velocity, is also studied.

1. Introduction

The interaction of waves and currents is of importance for a good prediction of the vertical structure of the mean flow field and the resulting morphodynamics in coastal areas. Processes that couple mean and fluctuating motions have been the subject of numerous publications. A large number of theoretical solutions for waves on currents with uniform or sheared profiles exist and have been discussed in a number of review articles, e.g. Peregrine (1976) or Jonsson (1990). In contrast to the large number of theoretical solutions, the amount of experimental data is very limited. The available data tend to concentrate on flow features in the near-bed region. At greater heights above the beds some experimental data have been reported by Bakker & Van Doorn (1978) and Kemp & Simons (1982, 1983). The latter considered waves following and opposing a turbulent current. Swan (1990) observed the modification of the wave motion in the presence of a strongly sheared current velocity throughout the depth of the flow field and compared these data with existing theoretical solutions. Klopman (1994) measured orbital and mean velocities over the whole depth of a channel in experiments concerning waves following and opposing a turbulent current.

Waves are known to have a considerable impact on the mean-velocity profiles. Longuet-Higgins (1953) emphasized the role of viscosity by showing that outside the thin oscillatory viscous boundary layers near the bottom and the free surface there is a mean drift velocity. Craik (1982*a*) extended this theory by considering temporally and spatially decaying waves and discussed the influence of surface contamination. Russell & Osorio (1957) and Mei, Liu & Carter (1972) measured drift velocities in a wave flume. Mei *et al.* (1972) state on pp. 151–152 that they obtain quantitative agreement (for a certain range of wave slopes kh) with Longuet-Higgins' conduction solution. More recently, Iskandarani & Liu (1991*a,b*, 1993) reported on experiments on two- and three-dimensional mass transport velocity profiles.

Waves not only induce mean velocities, but also modify existing current profiles. Experiments in laboratory channels like those of Bakker & Van Doorn (1978), Kemp & Simons (1982, 1983) and Klopman (1994), have indicated that the mean-velocity shear increases when waves propagate against the current direction and decreases or even changes sign when waves are propagating in the current direction. To the authors' knowledge, these observations have only been explained qualitatively, by Nielsen & You (1996) and Dingemans *et al.* (1996). The model proposed by Nielsen & You (1996) relies on a local force balance in a plane in the streamwise direction. Dingemans *et al.* (1996) show that secondary circulations in the cross-sectional plane, which are the result of the so-called wave-induced Craik–Leibovich (CL) vortex force, are responsible for changes in the mean horizontal velocities in the streamwise direction. However, for both models the results do not agree quantitatively with the experimental results.

The problems of understanding the mechanism of wave–current interactions have mostly been tackled via the Eulerian equations of mean motion. However, a major difficulty with the Eulerian representation of the flow field is the unique and unambiguous identification of the mean motion in an otherwise oscillating field, since at a fixed position between wave trough and wave crest there is water only part of the time. This difficulty can be avoided by considering the Lagrangian representation of the flow field. However, this formulation cannot be applied in any exact sense, if the Lagrangian-mean velocity is required at a specific point in space. This is due to the fact that the particle to be followed will generally wander away from this point. A consistent way to split the mean and oscillating motion is through the *generalized Lagrangian-mean* (GLM) method as proposed by Andrews & McIntyre (1978*a*). This is a hybrid Eulerian–Lagrangian description of motion, in which the Lagrangian-mean flow is described by means of equations in Eulerian form.

Andrews & McIntyre (1978*a*) derived the exact equations of GLM motion from the compressible Navier–Stokes equations. In these equations the wave forcing can be expressed either by the pseudomomentum or by a radiation-stress tensor, although the latter approach was merely used to provide a general framework for explaining the asymmetry of the radiation-stress tensors. In their two papers Andrews & McIntyre (1978*a,b*) concentrated on the influence of the pseudomomentum on the mean motion and its physical and conceptual meaning. Later, Grimshaw (1984) employed a technique based on variational principles to describe the wave–current interaction. Not only the pseudomomentum approach, often seen as characteristic for the GLM equations, but also the radiation-stress formulation were emphasized.

Despite the fact that Andrews & McIntyre (1978*a*) took dissipative forces into account, they were not treated in detail. Grimshaw (1981) included dissipation due to viscous effects in a GLM model which predicts the flow generated by a progressive non-breaking wave packet in otherwise still water. The assumption that an initial current is absent simplified the expressions for the dissipative forces significantly. Grimshaw (1981) produced analytical solutions for the GLM flow, up to second order in wave amplitude a.

The purpose of this paper is to develop a model which describes the flow field associated with non-breaking long-crested waves on an arbitrary current. Nonlinear changes in the mean velocity profile due to the presence of waves are considered, as well as modifications of the orbital velocity profiles caused by a current. Compared to Grimshaw's model the viscous terms turn out to be very lengthy and therefore difficult to implement in the model. For this reason an alternative approach to derive the GLM equations is described as well. This approach follows the radiation-stress approach mentioned above and is shown to bear some resemblance to the Eulerian equations of motion.

The arrangement of this paper is as follows. The GLM formulation of the equations describing the mean and fluctuating motion is given in § 2. The turbulence model that provides the closure of the flow equations if the motion is turbulent is discussed in § 3.1. The alternative approach to the GLM formulation is outlined in § 3.2. Proper boundary conditions at the bottom and the free surface are derived in § 4. The GLM equations are solved using a WKBJ-type perturbation-series approach. The resulting boundary-value problem, with only the vertical spatial coordinate as independent variable, is solved numerically. This solution method is described in § 5. In § 6 the presented model is applied to wave–current channel problems and the results from our model are compared with both theoretical solutions and experimental data. A summary and conclusions are finally given in § 7.

2. Generalized Lagrangian-mean formulation

Before proceeding with the description of the GLM theory, some remarks on the notation throughout this paper are made. Latin indices i, j or k take the value 1, 2 or 3. It is sometimes convenient in the subsequent analysis to distinguish between horizontal coordinates x_α ($\alpha = 1, 2$) and the vertical coordinate $z = x_3$ by employing Greek indices for horizontal variables and Latin variables for all three coordinates. Similarly, u_α denotes a horizontal velocity component and $w = u_3$ the vertical velocity. A different notation has been used for vectors operating in all three directions, $\boldsymbol{u} = (u_1, u_2, u_3)$ and those involving the horizontal direction, $\boldsymbol{u}_h = (u_1, u_2)$. Furthermore, Einstein's summation convention is used, i.e. $u_j v_j = u_1 v_1 + u_2 v_2 + u_3 v_3$.

The description of the basic formalism of the GLM theory is outlined here in a somewhat abstract way. In the GLM theory GLM operators are related to Eulerian-mean operators. Let $\overline{(\,)}$ be a general averaging operator (time, space, ensemble, etc.) taking a scalar, vector or tensor field $\varphi(\boldsymbol{x}, t)$ into a corresponding (Eulerian-mean) field $\overline{\varphi}(\boldsymbol{x}, t) = \overline{\varphi(\boldsymbol{x}, t)} = \langle \varphi(\boldsymbol{x}, t) \rangle$ at position $\boldsymbol{x}$ and time t. The notation $\langle (\,) \rangle$ is also used for the same averaging operator, for the sake of convenience. Whether the operator denotes time, space or ensemble averaging is not relevant for the general outline of the GLM theory at this stage. When model results from the GLM theory are compared with experimental data or when theoretical ideas are fixed, the averaging operation has to be specified.

To overcome the problems mentioned in § 1, arising in the Eulerian and pure Lagrangian framework, Andrews & McIntyre (1978*a*) generalized the classical Lagrangian-mean description in such a way that the expression 'Lagrangian-mean velocity *field*' makes sense. For the definition and notion of the GLM theory we could just refer to Andrews & McIntyre (1978*a*), or McIntyre (1980) for an introductory outline, but for completeness it will be repeated briefly here. An essential part in the GLM theory is the definition of the particle displacement $\boldsymbol{\xi}$ associated with the waves. Like all quantities in the GLM formulation, it is defined as a function of the position $\boldsymbol{x}$ and time t and no longer primarily as a function of the individual particle label as in a purely Lagrangian description. In fact, the generalized Lagrangian flow is described by means of equations in Eulerian form. When the disturbance-associated particle displacement $\boldsymbol{\xi}(\boldsymbol{x},t)$ has been defined, the exact GLM operator $\overline{(\ )}^L$, corresponding to any given Eulerian-mean operator $\langle(\ )\rangle$, can be defined as

$$\overline{\varphi(\boldsymbol{x},t)}^L = \langle \varphi(\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t),t)\rangle \,, \tag{2.1}$$

i.e. the average is taken with respect to the disturbed positions. Instead of $\varphi(\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t),t)$ the notation $\varphi^{\xi}(\boldsymbol{x},t)$ is used as well. φ^{ξ} is also called the shifted φ. Furthermore, the disturbed position is denoted by $\Xi(\boldsymbol{x},t)=\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)$.

By assuming that the mapping $\boldsymbol{x}\rightarrow\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)$ is invertible, Andrews & McIntyre (1978*a*, p. 615) stated that for any given $\boldsymbol{u}(\boldsymbol{x},t)$ there is a unique 'reference velocity field' $\boldsymbol{v}(\boldsymbol{x},t)$, such that, when the point $\boldsymbol{x}$ moves with the velocity $\boldsymbol{v}$ the point $\boldsymbol{x}+\boldsymbol{\xi}$ moves with the actual velocity $\boldsymbol{u}^{\xi}$, i.e.

$$\left(\partial/\partial t + v_j\partial/\partial x_j\right)\Xi = \boldsymbol{u}^{\xi}. \tag{2.2}$$

Andrews & McIntyre (1978*a*, p. 615) obtained the GLM description by requiring that

$$\overline{\boldsymbol{\xi}(\boldsymbol{x},t)} = 0, \tag{2.3}$$

i.e. that $\boldsymbol{\xi}$ is a true disturbance quantity and that the reference velocity $\boldsymbol{v}$ is a mean quantity,

$$\boldsymbol{v}(\boldsymbol{x},t) = \overline{\boldsymbol{v}(\boldsymbol{x},t)}, \tag{2.4}$$

which yields $\boldsymbol{v}=\overline{\boldsymbol{u}}^L$. By introducing the Lagrangian disturbance velocity $\boldsymbol{u}^{\ell}$ in a natural way as

$$\boldsymbol{u}^{\ell}(\boldsymbol{x},t) = \boldsymbol{u}^{\xi}(\boldsymbol{x},t) - \overline{\boldsymbol{u}(\boldsymbol{x},t)}^L \tag{2.5}$$

relation (2.2) yields

$$\overline{\mathrm{D}}^L\boldsymbol{\xi} = \boldsymbol{u}^{\ell}, \tag{2.6}$$

with $\overline{\mathrm{D}}^L = \partial/\partial t + \overline{u}_j^L\partial/\partial x_j$ the generalized Lagrangian-mean material derivative, denoting the rate of change following the GLM flow. Relation (2.6) between the disturbance-associated fields $\boldsymbol{\xi}$ and $\boldsymbol{u}^{\ell}$ not only defines $\boldsymbol{\xi}$ but also validates the claim to regard $\boldsymbol{\xi}$ as a disturbance-associated particle displacement.

Finally, Eulerian-mean and generalized Lagrangian-mean quantities are related to each other by the generalized 'Stokes correction' $\overline{\varphi}^S$, which is defined as (see also Appendix A, § A.2 for an expression in terms of GLM quantities)

$$\overline{\varphi(\boldsymbol{x},t)}^S = \overline{\varphi(\boldsymbol{x},t)}^L - \overline{\varphi(\boldsymbol{x},t)}. \tag{2.7}$$

When φ denotes velocity $\boldsymbol{u}$, the Stokes correction $\overline{\boldsymbol{u}}^S$ is sometimes referred to as Stokes drift.

The derivation of the GLM equations of motion is based on the equations of motion for an incompressible homogeneous fluid in an Eulerian formulation, which are given by the mass-conservation equation,

$$\frac{\partial u_j}{\partial x_j} = 0, \tag{2.8}$$

and the momentum equation,

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} - X_i = F_i, \tag{2.9}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + u_j\partial/\partial x_j$ denotes the material derivative, p is ressure and ρ the density. $\boldsymbol{X}$ is a function which allows for contributions which can be ascribed to viscosity and/or turbulence and whose form is left general is this section. In § 3 the function $\boldsymbol{X}$ will be specified. Finally, $\boldsymbol{F}$ represents large-scale driving forces for the mean flow in the horizontal direction and equals the gravitational acceleration in the vertical direction. These equations are completed with boundary conditions at the bottom and free surface, which are derived in § 4.

Andrews & McIntyre (1978*a*) derived the exact GLM equations of motion from the compressible Navier–Stokes equations. Nevertheless, the GLM theory can also be applied to incompressible flow problems. The general idea is to consider the variables in (2.8), (2.9) at their displaced positions, multiplying the momentum equations by $\partial\Xi_j/\partial x_i$ and taking the mean of the resulting equation. These operations result in the GLM equations which are given by

$$\frac{\partial \overline{u}_j^L}{\partial x_j} = -\overline{\mathrm{D}}^L(\log J), \tag{2.10}$$

and

$$\overline{\mathrm{D}}^L\overline{u}_i^L + \frac{1}{\rho}\frac{\partial \overline{p}^L}{\partial x_i} - \overline{X}_i^L - \overline{F}_i^L$$
$$= \left\langle \frac{\partial \xi_j}{\partial x_i} X_j^\ell \right\rangle + \left\langle \frac{\partial \xi_j}{\partial x_i} F_j^\ell \right\rangle + \overline{\mathrm{D}}^L\overline{P}_i^L + \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}\left(\overline{u_j^\ell u_j^\ell}\right)\right) + \overline{P}_j^L\frac{\partial \overline{u}_j^L}{\partial x_i}. \tag{2.11}$$

Here J is the Jacobian of the mapping $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{\xi}$, i.e. $J = \det(\partial\Xi_j/\partial x_i)$, and $\overline{P}_i^L = -\langle u_j^\ell \partial\xi_j/\partial x_i\rangle$ is the so-called pseudomomentum. For an extensive derivation of the GLM flow equations (2.10), (2.11) reference is made to Andrews & McIntyre (1978*a*) and Dingemans (1997, § 2.10.6). The influence of the wave motion on the mean motion is given by the right-hand side of (2.11). In order to quantify the wave-induced force on the mean motion the pseudomomentum seems an important quantity. For the physical and conceptual meaning of the pseudomomentum see e.g. Andrews & McIntyre (1978*b*), Craik (1982*b*) or Grimshaw (1984).

The GLM equations of motions are exact equations in the sense that no asymptotic analysis is required as long as viscous or turbulence effects are not fully specified and left as general as in (2.11). As Andrews & McIntyre (1978*a*) mentioned throughout their derivation of the GLM equations, these equations also hold for finite-amplitude waves. Incorporation of viscous effects unavoidably requires some asymptotic analysis. A correct formulation of the viscous part in terms of GLM quantities leads to lengthy expressions for the function $\overline{X}_i^L$. As to be expected the situation is worse for turbulent motion. Therefore, the derivation of the GLM equations by Andrews & McIntyre (1978*a*) is adapted in § 3.2.

3. A form of the GLM equations analogous to the Eulerian mean equations

Before proceeding with the alternative derivation of the GLM equations of motion, the function $\boldsymbol{X}$ is specified in order to deal with viscosity or turbulence. When the flow is considered viscous and non-turbulent, the function $\boldsymbol{X}$ can be expressed using the stress tensor,

$$X_i = \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}, \quad \frac{\tau_{ij}}{\rho} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3.1}$$

where ν equals the kinematic viscosity. In fact, the same representation can be used in the turbulent regime. The shear stresses are directly related to the strain-rate tensor by using Boussinesq's hypothesis. A turbulence model has been implemented to determine the eddy viscosity ν and provide a proper closure of the model equations.

3.1. *Turbulence model*

A review of turbulence models and their use in hydrodynamic problems can be found in Rodi (1984). Following Klopman (1992) a one-equation turbulence model has been implemented. By denoting the turbulence kinetic energy per unit of mass by q, the evolution equation of q can be modelled as

$$\frac{\mathrm{D}q}{\mathrm{D}t} - \mathscr{P} + \mathscr{D} - \frac{\partial f_j}{\partial x_j} = 0, \tag{3.2}$$

with

$$\mathscr{P} \equiv \nu \left(\frac{\partial u_m}{\partial x_k} + \frac{\partial u_k}{\partial x_m} \right) \frac{\partial u_k}{\partial x_m}, \tag{3.3}$$

$$\mathscr{D} \equiv C_D \frac{q^{3/2}}{\ell}, \tag{3.4}$$

$$f_i \equiv \frac{\nu}{\sigma_k} \frac{\partial q}{\partial x_i}. \tag{3.5}$$

Here $\mathscr{P}, \mathscr{D}, f_i$ denote the turbulence kinetic energy production, dissipation and flux in the x_i-direction respectively. Furthermore, ℓ is the prescribed turbulence lengthscale and C_D and σ_k are empirical constants. The eddy viscosity is related to q and ℓ by

$$\nu = C_\mu' q^{1/2} \ell, \tag{3.6}$$

with C_μ' another constant.

In § 5.2 the implementation of this turbulence model in the GLM model will be outlined in more detail. Values for the empirical constants as well as an expression for the mixing-length will be given in § 6.

3.2. *Inclusion of shear stresses in GLM equations*

The GLM counterpart $\overline{\boldsymbol{X}}^L$ can be expressed in terms of GLM quantities by averaging (3.1) at disturbed positions. Second-order partial derivatives of the velocities lead to very lengthy expressions. They will be omitted in this paper. Another option is to split the transformation of $\boldsymbol{X}$ into two steps in which the shear stress tensor τ_{ij} is maintained as dependent variable. However, evaluation at the disturbed positions still requires a lot of effort. In order to obtain lucid expressions for $\overline{\boldsymbol{X}}^L$, the equations are no longer evaluated at the disturbed positions, but at the fixed positions. As mentioned in the introduction Andrews & McIntyre (1978 *a*, § 8) used this approach as well to show the general limitations of the 'radiation stress' concept.

As already stated, the momentum equation (2.9) is considered at the fixed point $\boldsymbol{x}$. In Appendix A, §§ A.1 to A.3 the GLM equations are derived in an alternative way, using asymptotic analysis at some stage. Here only the resulting equations are stated. The disturbance-related quantities, like $\boldsymbol{\xi}$ and $\boldsymbol{u}^\ell$, scale with the wave motion, which has amplitude a. This wave amplitude is supposed to be small with respect to both depth and wavelength. Despite the fact that a is not dimensionless, the order of approximation is denoted by $O(a^n)$. Since the equations of motion are not non-dimensionalized, $O(a)$ is used for convenience instead of $O(\epsilon)$ with $\epsilon = ka$ or $\epsilon = a/h$.

Application of several properties of the mapping $\boldsymbol{x} \rightarrow \boldsymbol{x} + \boldsymbol{\xi}$ and Taylor series expansion of the total motion around the disturbed position $\boldsymbol{x} + \boldsymbol{\xi}$ leads, after averaging, to equations of motion for the mean motion. As shown in Appendix A, § A.3, subtraction of the equation for the mean motion from the equation for the total motion result in an equation for the fluctuating part,

$$\overline{\mathrm{D}}^L u_i^\ell + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p^\ell - \xi_j \frac{\partial \overline{p}^L}{\partial x_j}\right) - \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}^\ell - \xi_k \frac{\partial \overline{\tau}_{ij}^L}{\partial x_k}\right) = \xi_j \frac{\partial}{\partial x_j}\left(\overline{\mathrm{D}}^L \overline{u}_i^L\right) + O(a^2). \quad (3.7)$$

The disturbance displacement $\boldsymbol{\xi}$ is related to the disturbance velocity $\boldsymbol{u}^\ell$ by relation (2.6). For the mean motion the following equation holds:

$$\overline{\mathrm{D}}^L \overline{u}_i^L + \frac{1}{\rho}\frac{\partial \overline{p}^L}{\partial x_i} - \frac{1}{\rho}\frac{\partial \overline{\tau}_{ij}^L}{\partial x_j} - \overline{F}_i^L = \overline{S}_i^L. \quad (3.8)$$

The wave-induced driving force for the mean motion $\overline{S}_i^L$ can be written as

$$\overline{S}_i^L = \frac{1}{\rho}\frac{\partial \overline{p}^S}{\partial x_i} - \frac{1}{\rho}\frac{\partial \overline{\tau}_{ij}^S}{\partial x_j} - \overline{F}_i^S + \overline{R}_i^L, \quad (3.9)$$

where

$$\overline{R}_i^L = -\frac{\partial r_{ij}}{\partial x_j} - \left\langle \xi_j \frac{\partial \xi_k}{\partial x_j} \right\rangle \frac{\partial}{\partial x_k}\left(\overline{\mathrm{D}}^L \overline{u}_i^L\right) - \tfrac{1}{2}\overline{\xi_j \xi_k}\frac{\partial^2}{\partial x_j \partial x_k}\left(\overline{\mathrm{D}}^L \overline{u}_i^L\right) + O(a^3), \quad (3.10)$$

and

$$r_{ij} = \overline{u_i^\ell u_j^\ell} - \overline{\mathrm{D}}^L\left(\overline{u_i^\ell \xi_j}\right). \quad (3.11)$$

In Appendix A, § A.4 expressions are derived which relate the mean and fluctuating shear stresses $\overline{\tau}_{ij}^L$ and τ_{ij}^ℓ to the GLM and disturbed velocities for arbitrary distributions of the eddy viscosity ν.

The tensor r_{ij} is closely related to the radiation-stress tensors defined, amongst others, by Longuet–Higgins & Stewart (1964). Andrews & McIntyre (1978*a*, p. 634), who derived a tensor which depends on the pressure part of the driving force $\overline{S}_i^L$ in (3.9), stated that in order to be called a radiation stress, r_{ij} must represent the sole effect of the waves on the mean flow. Although $\overline{R}_i^L$ will be dominated by r_{ij}, for turbulent motion the effect of the Stokes correction of the shear stress $\overline{\tau}_{ij}^S$ on the wave-induced driving force $\overline{S}_i^L$ will be significant. In the case of inviscid motion, Andrews & McIntyre concluded that r_{ij} represents the sole effect of the waves and thus can be called a radiation stress.

When viscous and/or turbulence effects are neglected, the GLM equations consisting of the mass conservation equation (2.10) and the momentum equation (3.8) are not exact, i.e. third- and higher-order wave-induced terms have been omitted. In contrast to the original GLM momentum equation (2.11) the alternative GLM equations have to be modified if more accurate solutions are required. Therefore, the alternative GLM equations do not seem to be convenient for inviscid flows. However, if the shear stresses have to be included, asymptotic analysis will lead to descriptions which are more lucid in the alternative approach. Moreover, the function X_i is written as a divergence of the shear stresses in order to obtain a system of first-order differential equations. Furthermore, in the form presented in this section the GLM equations have a similar form to the Eulerian equations. By evaluating the right-hand side of equations (2.10) and (3.8) the GLM quantities can be computed by existing numerical models which solve the Eulerian equations (2.8), (2.9). Finally, boundary conditions are often formulated in terms of normal and tangential stresses. In the present formulation, in which the shear stresses are dependent variables, these conditions can be worked out easily, as will be shown in § 4.

4. Boundary conditions

An important aspect in this analysis is the derivation of the boundary conditions, especially at the free surface. The bottom is assumed to be impermeable, resulting in a vanishing vertical velocity at the bottom. Furthermore, the bottom is a so-called no-slip boundary, causing the horizontal velocity to vanish. Therefore the following conditions hold at $z = -h(\boldsymbol{x}_h)$:

$$u_i = 0. \tag{4.1}$$

At the bottom the production of turbulence kinetic energy is assumed to be in balance with the dissipation of turbulence kinetic energy. This results in the following boundary condition:

$$q - \frac{v}{(C'_\mu)^2}\left[\left(\frac{\partial u_k}{\partial x_m} + \frac{\partial u_m}{\partial x_k}\right)\frac{\partial u_k}{\partial x_m}\right]^{1/2} = 0. \tag{4.2}$$

Andrews & McIntyre (1978*a*) have mentioned that at an impermeable boundary the component of the GLM velocity normal to the bottom boundary equals the velocity of the boundary itself. Moreover, since a particle at the bottom will stick to its position the disturbance displacement and so the GLM velocity will vanish at the bottom,

$$\xi_i = 0, \quad \overline{u}_i^L = 0. \tag{4.3}$$

At the free surface, which is unknown beforehand, two types of boundary conditions are imposed. The kinematic boundary condition states that the normal velocity component of the free surface equals the normal velocity component of the flow,

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = w. \tag{4.4}$$

The dynamic boundary condition denotes a balance between the normal and shear stresses on both sides of the free surface,

$$n_i\tau_{ij}n_j - p = -p_F, \tag{4.5a}$$

$$n_i\tau_{ij}r_j = -\tau_{F_r}, \tag{4.5b}$$

$$n_i\tau_{ij}s_j = -\tau_{F_s}, \tag{4.5c}$$

where $\boldsymbol{n}, \boldsymbol{r}, \boldsymbol{s}$ form an orthonormal set of vectors, such that the $\boldsymbol{n}$-direction is normal to the free surface and the $\boldsymbol{r}$- and $\boldsymbol{s}$-directions are tangential to the free surface. In (4.5) surface tension effects are neglected. Moreover, p_F denotes the pressure and τ_{F_r}, τ_{F_s} equal the wind shear stress components just above the free surface. The boundary condition for the turbulence kinetic energy is assumed to be a symmetry condition (see Klopman 1992):

$$\frac{\partial q}{\partial \boldsymbol{n}} = \frac{\partial q}{\partial x_j}n_j = f_j n_j = 0. \tag{4.6}$$

The free-surface boundary conditions are transformed into a GLM formulation without special effort, if the assumption that the mapping $\boldsymbol{x} \rightarrow \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t)$ is invertible is accepted. If the fluid particles are assumed to be in their disturbed positions Ξ, one can always split these positions as $\Xi = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t)$. Hence, if the free surface in Eulerian coordinates is given by

$$z - \zeta(\boldsymbol{x}_h, t) = 0, \tag{4.7}$$

with ζ the deviation of the free surface from the still-water level $z = 0$, and a point on the free surface is considered to be in a disturbed position, then by splitting the disturbed position, one finds

$$z + \xi_3(\boldsymbol{x}_h, z, t) - \zeta(\boldsymbol{x}_h + \boldsymbol{\xi}_h(\boldsymbol{x}_h, z, t), t) = 0. \tag{4.8}$$

According to definitions (2.1) and (2.3) averaging relation (4.8) results in a description of the free surface in GLM coordinates,

$$z = \overline{\zeta}^L(\boldsymbol{x}_h, t). \tag{4.9}$$

Furthermore, at this level the vertical displacement equals the oscillating part of the GLM free-surface elevation, i.e.

$$\eta(\boldsymbol{x}_h, z, t) \equiv \xi_3(\boldsymbol{x}_h, z, t) = \zeta^l(\boldsymbol{x}_h, t) \quad \text{for} \quad z = \overline{\zeta}^L(\boldsymbol{x}_h, t). \tag{4.10}$$

This approach was followed by Grimshaw (1981) as well. By applying relations (4.8) and (4.9) the following identity can be derived:

$$\begin{aligned} &\varphi^{\xi}(\,\boldsymbol{x}_h,\, \overline{\zeta}^L(\boldsymbol{x}_h, t),\, t\,) \\ &\quad \equiv \varphi\left\{\, \boldsymbol{x}_h + \boldsymbol{\xi}_h\left[\boldsymbol{x}_h, \overline{\zeta}^L(\boldsymbol{x}_h, t), t\right],\; \overline{\zeta}^L(\boldsymbol{x}_h, t) + \eta\left[\boldsymbol{x}_h, \overline{\zeta}^L(\boldsymbol{x}_h, t), t\right],\; t \,\right\} \\ &\quad = \varphi\left\{\, \boldsymbol{x}_h + \boldsymbol{\xi}_h\left[\boldsymbol{x}_h, \overline{\zeta}^L(\boldsymbol{x}_h, t), t\right],\; \zeta\left(\boldsymbol{x}_h + \boldsymbol{\xi}_h\left[\boldsymbol{x}_h, \overline{\zeta}^L(\boldsymbol{x}_h, t), t\right],\; t\right),\; t\right\}, \end{aligned} \tag{4.11}$$

which implies that generalized Lagrangian arguments are attached to the free surface in a GLM formulation if Eulerian arguments are evaluated at the free surface in an Eulerian framework.

By applying relation (4.11) the kinematic boundary condition (4.4) is transformed directly to

$$\overline{\mathrm{D}}^L \zeta^{\xi} = w^{\xi} \quad \text{for} \quad z = \overline{\zeta}^L(\boldsymbol{x}_h, t). \tag{4.12}$$

If the pressure and the wind shear stresses just above the free surface are neglected, the dynamic boundary conditions (4.5) are equivalent to

$$-p n_i + \tau_{ij} n_j = 0 \quad \text{for} \quad z = \zeta(\boldsymbol{x}_h, t), \tag{4.13}$$

with $n_\alpha = -\partial\zeta/\partial x_\alpha$, $n_3 = 1$. Application of relation (4.11) immediately yields

$$-p^{\xi} n_i^{\xi} + \tau_{ij}^{\xi} n_j^{\xi} = 0 \quad \text{for} \quad z = \overline{\zeta}^L(\boldsymbol{x}_h, t), \tag{4.14}$$

with

$$n_\alpha^{\xi} = -\left(\frac{\partial \zeta^{\xi}}{\partial x_a} - \frac{\partial \xi_\beta}{\partial x_a}\frac{\partial \zeta^{\xi}}{\partial x_b}\right) + O(a^3), \quad n_3^{\xi} = 1. \tag{4.15}$$

After averaging, proper boundary conditions at the free surface $z = \overline{\zeta}^L(\boldsymbol{x}, t)$ are obtained. For the GLM flow they read up to second order

$$\overline{\tau}^L_{\alpha 3} = -\left\langle p^\ell \frac{\partial \zeta^\ell}{\partial x_a} \right\rangle - \overline{p}^L \left(\frac{\partial \overline{\zeta}^L}{\partial x_a} - \left\langle \frac{\partial \xi_\beta}{\partial x_a} \frac{\partial \zeta^\ell}{\partial x_b} \right\rangle \right) + \left\langle \tau^\ell_{\alpha\beta} \frac{\partial \zeta^\ell}{\partial x_b} \right\rangle + \overline{\tau}^L_{\alpha\beta} \left(\frac{\partial \overline{\zeta}^L}{\partial x_b} - \left\langle \frac{\partial \xi_\gamma}{\partial x_b} \frac{\partial \zeta^\ell}{\partial x_\gamma} \right\rangle \right) + O(a^3), \tag{4.16a}$$

$$-\overline{p}^L + \overline{\tau}^L_{33} = \left\langle \tau^\ell_{3\beta} \frac{\partial \zeta^\ell}{\partial x_b} \right\rangle + \overline{\tau}^L_{3\beta} \left(\frac{\partial \overline{\zeta}^L}{\partial x_b} - \left\langle \frac{\partial \xi_\gamma}{\partial x_b} \frac{\partial \zeta^\ell}{\partial x_\gamma} \right\rangle \right) + O(a^3), \tag{4.16b}$$

and for the fluctuating motion up to first order,

$$\tau^\ell_{\alpha 3} = -\overline{p}^L \frac{\partial \zeta^\ell}{\partial x_a} + \overline{\tau}^L_{\alpha\beta} \frac{\partial \zeta^\ell}{\partial x_b} + O(a^2), \tag{4.17a}$$

$$-p^\ell + \tau^\ell_{33} = \overline{\tau}^L_{3\beta} \frac{\partial \zeta^\ell}{\partial x_b} + O(a^2). \tag{4.17b}$$

5. WKBJ perturbation-series approach

The equations for the mean and fluctuating motion show that the wave motion has an impact on the mean motion and vice versa. The equations describing both types of motion can be solved simultaneously but this would be very inefficient due to nonlinearities. Therefore, a WKBJ-type expansion into perturbation series is carried out to distinguish between the slow modulation of the current profile in time and the horizontal direction and the fast varying wave components.

5.1. *Description of the WKBJ method*

The essence of the WKBJ-expansion method is to suppose that the amplitude function A of a quantity $\varphi(\boldsymbol{x}, t)$ varies much more slowly in time and horizontal space than the phase function S. We suppose that φ can be represented as

$$\varphi(\boldsymbol{x}, t) = \varphi(\boldsymbol{x}_h, z, t) = A(\boldsymbol{x}_h, z, t)\ \exp\left(\mathrm{i}S(\boldsymbol{x}_h, z, t)\right), \tag{5.1}$$

with $\mathrm{i} = \sqrt{-1}$. Let δ be a small modulation parameter, indicating the slight relative variation in the mean motion on the scale of the characteristic wavelength. By introducing slowly varying temporal and horizontal spatial coordinates

$$\boldsymbol{X}_h = \delta \boldsymbol{x}_h, \quad T = \delta t, \tag{5.2}$$

the function φ in (5.1) can be rewritten as

$$\varphi(\boldsymbol{x}, t) = A(\boldsymbol{X}_h, z, T)\ \exp\left(\mathrm{i}S(\boldsymbol{x}_h, z, t)/\delta\right). \tag{5.3}$$

More details on the WKBJ expansion method can be found e.g. in Olver (1974, chapter 6).

A variation on the WKBJ method is given by Chu & Mei (1970). They introduced the characteristic wave slope $\epsilon = ka$, with k and a characteristic values of the wavenumber and the wave amplitude, as a nonlinearity parameter and assumed ϵ to be of the same order as the modulation parameter δ. By expanding to both nonlinearity and rate of modulation the following expansions of WKBJ type were assumed:

$$\varphi(\boldsymbol{x},t)=\sum_{n=0}^{\infty}\epsilon^n\sum_{m=-n}^{+n}\hat{\varphi}^{(n,m)}(\boldsymbol{X}_h,z,T)E^m, \tag{5.4}$$

where

$$E=\exp\left(\mathrm{i}\left(k_\beta X_\beta-\omega T\right)/\epsilon\right). \tag{5.5}$$

with $\hat{\varphi}^{(n,-m)}$ complex conjugates of the amplitude functions $\hat{\varphi}^{(n,m)}$. The wavenumber and wave frequency are denoted by $\boldsymbol{k}$ and ω. Chu & Mei (1970) expanded $\boldsymbol{k}$ and ω to nonlinearity as well, which is omitted in this paper. The approach of expanding each quantity, except the wavenumber and frequency, into the perturbation series given by (5.4), has been used before by Lo & Mei (1985).

5.2. *Application of the WKBJ method to the GLM equations*

Due to the introduction of slow horizontal and temporal coordinates, substitution of variables as perturbation series into the GLM equations of motion (2.10), (3.8) results in a cascade of problems at the different orders of approximation, which can be solved successively. These problems are systems of ordinary differential equations (ODEs) with the vertical coordinate z as the only independent variable. Except for the $(0,0)$-problem, these ODEs are linear. In the obtained hierarchical system the (0,0)-solution is the basic solution, describing a uniform steady current. Slow variations are described by the $(1,0)$-solution. The $(1,1)$-solution is the wave part, describing the motion of the waves according to the linear theory including the effect of mean velocity shear. Finally, the $(2,0)$-solution describes the second-order changes in the mean velocity profile due to the presence of waves.

Substitution of the expanded forms into the governing equations (2.10), (3.8) and boundary conditions at the bottom boundary (4.3) and at the free surface (4.12), (4.14), results in the following set of ODEs for each index (n,m):

$$\mathrm{i}mk_\beta\hat{u}_\beta^{(n,m)}+\frac{\partial\hat{w}^{(n,m)}}{\partial z}=\hat{F}^{(n,m)}, \tag{5.6}$$

$$-\mathrm{i}m\omega_0\hat{u}_\alpha^{(n,m)}+\hat{w}^{(0,0)}\frac{\partial\hat{u}_\alpha^{(n,m)}}{\partial z}+\mathrm{i}mk_\alpha\frac{1}{\rho}\left(\hat{p}^{(n,m)}-\hat{\eta}^{(n,m)}\frac{\partial\hat{p}^{(0,0)}}{\partial z}\right)$$
$$-\mathrm{i}mk_\beta\frac{1}{\rho}\left(\hat{\tau}_{\alpha\beta}^{(n,m)}-\hat{\eta}^{(n,m)}\frac{\partial\hat{\tau}_{\alpha\beta}^{(0,0)}}{\partial z}\right)-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\hat{\tau}_{\alpha3}^{(n,m)}-\hat{\eta}^{(n,m)}\frac{\partial\hat{\tau}_{\alpha3}^{(0,0)}}{\partial z}\right)=\hat{G}_\alpha^{(n,m)}, \tag{5.7a}$$

$$-\mathrm{i}m\omega_0\hat{w}^{(n,m)}+\hat{w}^{(0,0)}\frac{\partial\hat{w}^{(n,m)}}{\partial z}+\frac{1}{\rho}\frac{\partial}{\partial z}\left(\hat{p}^{(n,m)}-\hat{\eta}^{(n,m)}\frac{\partial\hat{p}^{(0,0)}}{\partial z}\right)$$
$$-\mathrm{i}mk_\beta\frac{1}{\rho}\left(\hat{\tau}_{3\beta}^{(n,m)}-\hat{\eta}^{(n,m)}\frac{\partial\hat{\tau}_{3\beta}^{(0,0)}}{\partial z}\right)-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\hat{\tau}_{33}^{(n,m)}-\hat{\eta}^{(n,m)}\frac{\partial\hat{\tau}_{33}^{(0,0)}}{\partial z}\right)=\hat{G}_3^{(n,m)}. \tag{5.7b}$$

Here $\omega_0=\omega-k_\beta\hat{u}_\beta^{(0,0)}$ denotes the intrinsic frequency. The forcing functions $\hat{F}^{(n,m)}$, $\hat{G}_i^{(n,m)}$ are in terms of amplitude functions of order lower than n. Since $\overline{\eta(\boldsymbol{x},t)}=0$ due to restriction (2.3), the zeroth harmonic ($m=0$) of η equals zero

$$\hat{\eta}^{(n,0)}=0\quad\text{for}\quad n\geqslant 0, \tag{5.8}$$

and because the still-water level is described by $z=0$ not only in an Eulerian framework but in a GLM formulation as well, the zeroth-order free-surface elevation vanishes,

$$\hat{\zeta}^{(0,0)} = 0. \tag{5.9}$$

The amplitude function of the disturbance displacement $\boldsymbol{\xi}$ satisfies upon using (2.6),

$$-\mathrm{i}m\omega_0\hat{\xi}_i^{(n,m)} + \hat{w}^{(0,0)}\frac{\partial\hat{\xi}_i^{(n,m)}}{\partial z} = \hat{u}_i^{(n,m)} + \hat{D}_i^{(n,m)} \quad \text{for} \quad n \geqslant 1, m \neq 0, \tag{5.10}$$

with the bottom boundary condition

$$\hat{\xi}_i^{(n,m)} = 0 \quad \text{for} \quad z = -h \quad \text{and} \quad n \geqslant 1, m \neq 0. \tag{5.11}$$

Furthermore, at the free surface the vertical disturbance displacement equals the fluctuating part of the surface elevation, according to (4.10):

$$\hat{\eta}^{(n,m)} = \hat{\zeta}^{(n,m)} \quad \text{for} \quad z = \overline{\zeta}^L \quad \text{and} \quad n \geqslant 1, m \neq 0. \tag{5.12}$$

The boundary conditions at the bottom boundary are given by

$$\hat{u}_i^{(n,m)} = \hat{H}_i^{(n,m)}. \tag{5.13}$$

The boundary conditions at the free surface (4.12), (4.14) may be expanded into Taylor's series about $z = 0$,

$$\sum_{k=0}^{\infty} \frac{\left(\overline{\zeta}^L\right)^k}{k!} \frac{\partial^k}{\partial z^k}\left\{\overline{\mathrm{D}}^L \zeta^\zeta - w^\zeta\right\} = 0, \tag{5.14a}$$

$$\sum_{k=0}^{\infty} \frac{\left(\overline{\zeta}^L\right)^k}{k!} \frac{\partial^k}{\partial z^k}\left\{-p^\zeta n_i^\zeta + \tau_{ij}^\zeta n_j^\zeta\right\} = 0, \tag{5.14b}$$

resulting into the following conditions at $z = 0$ for each index (n, m):

$$-\mathrm{i}m\omega_0\hat{\zeta}^{(n,m)} - \hat{w}^{(n,m)} = \hat{L}^{(n,m)}, \tag{5.15a}$$

$$\hat{\tau}_{\alpha 3}^{(n,m)} + \hat{\zeta}^{(n,m)}\left(\mathrm{i}mk_\alpha\hat{p}^{(0,0)} - \mathrm{i}mk_\beta\hat{\tau}_{\alpha\beta}^{(0,0)}\right) + \delta(m)\hat{\zeta}^{(n,0)}\frac{\partial\hat{\tau}_{\alpha 3}^{(0,0)}}{\partial z} = \hat{K}_\alpha^{(n,m)}, \tag{5.15b}$$

$$-\hat{p}^{(n,m)} + \hat{\tau}_{33}^{(n,m)} - \mathrm{i}mk_\beta\hat{\zeta}^{(n,m)}\hat{\tau}_{3\beta}^{(0,0)} + \delta(m)\hat{\zeta}^{(n,0)}\left(-\frac{\partial\hat{p}^{(0,0)}}{\partial z} + \frac{\partial\hat{\tau}_{33}^{(0,0)}}{\partial z}\right) = \hat{K}_3^{(n,m)}. \tag{5.15c}$$

where $\delta(m) = 1$ for $m = 0$, otherwise $\delta(m) = 0$.

For the shear stresses the following expressions have been derived from relation (A 16):

$$\frac{\hat{\tau}_{\alpha\beta}^{(n,m)}}{\rho} = \hat{\nu}^{(0,0)}\left(\mathrm{i}mk_\beta\hat{u}_\alpha^{(n,m)} + \mathrm{i}mk_\alpha\hat{u}_\beta^{(n,m)}\right) + \hat{T}_{\alpha\beta}^{(n,m)}, \tag{5.16a}$$

$$\frac{\hat{\tau}_{\alpha 3}^{(n,m)}}{\rho} = \hat{\nu}^{(0,0)}\left(\frac{\partial\hat{u}_\alpha^{(n,m)}}{\partial z} + \mathrm{i}mk_\alpha\hat{w}^{(n,m)}\right) + \hat{T}_{\alpha 3}^{(n,m)}, \tag{5.16b}$$

$$\frac{\hat{\tau}_{33}^{(n,m)}}{\rho} = 2\,\hat{\nu}^{(0,0)}\frac{\partial\hat{w}^{(n,m)}}{\partial z} + \hat{T}_{33}^{(n,m)}. \tag{5.16c}$$

For convenience, the expressions for the right-hand sides $\hat{F}^{(n,m)}$, $\hat{G}_i^{(n,m)}$, $\hat{D}_i^{(n,m)}$, $\hat{H}_i^{(n,m)}$, $\hat{K}_i^{(n,m)}$, $\hat{L}^{(n,m)}$, $\hat{T}_{ij}^{(n,m)}$ are given in Appendix B up to second order.

The determination of the eddy viscosity ν is simplified. The distribution of ν is assumed to consist only of the basic component $\hat{\nu}^{(0,0)}$. This means that ν is independent of both time and wave motion and thus determined only by the steady current. Hence, after substituting the WKBJ-type expansions into equations (3.2)–(3.6) and corresponding boundary conditions (4.2), (4.6), which describe the q–ℓ turbulence model, only the resulting zeroth-order equations have to be considered. These nonlinear ODEs are given by

$$\hat{w}^{(0,0)}\frac{\partial \hat{q}^{(0,0)}}{\partial z} - \hat{\mathscr{P}}^{(0,0)} + \hat{\mathscr{D}}^{(0,0)} - \frac{\partial \hat{f}_3^{(0,0)}}{\partial z} = 0, \tag{5.17}$$

with

$$\hat{\mathscr{P}}^{(0,0)} \equiv \hat{\nu}^{(0,0)}\frac{\partial \hat{u}_\beta^{(0,0)}}{\partial z}\frac{\partial \hat{u}_\beta^{(0,0)}}{\partial z}, \tag{5.18a}$$

$$\hat{\mathscr{D}}^{(0,0)} \equiv C_D\frac{\left(\hat{q}^{(0,0)}\right)^{3/2}}{\ell}, \tag{5.18b}$$

$$\hat{f}_3^{(0,0)} \equiv \frac{\hat{\nu}^{(0,0)}}{\sigma_k}\frac{\partial \hat{q}^{(0,0)}}{\partial z}, \tag{5.18c}$$

$$\hat{\nu}^{(0,0)} = C_\mu^{'}\left(\hat{q}^{(0,0)}\right)^{1/2}\ell, \tag{5.18d}$$

and

$$\hat{q}^{(0,0)} = \frac{\hat{\nu}^{(0,0)}}{(C_\mu^{'})^2}\left(\frac{\partial \hat{u}_\beta^{(0,0)}}{\partial z}\frac{\partial \hat{u}_\beta^{(0,0)}}{\partial z}\right)^{1/2} \quad \text{for} \quad z=-h, \tag{5.19a}$$

$$\hat{f}_3^{(0,0)} = 0 \quad \text{for} \quad z = 0\,. \tag{5.19b}$$

The mixing length has to be specified. According to Rodi (1984) the turbulence lengthscale profile should be linear close to the bottom, i.e. $\ell(s) = \kappa s$, with s the distance to the bed and $\kappa = 0.41$ the von Kàrmàn constant. The choice for the mixing length ℓ, which is prescribed as function of the flow geometry only, of

$$\ell(z) = \kappa(z+h+z_0)\left(\frac{-z+z_a}{h+z_a}\right)^{1/2} \quad \text{for} \quad -h \leqslant z \leqslant 0, \tag{5.20}$$

was originally proposed by Bakhmetev (1932) for $z_a = 0$. The parameter z_a is introduced to ensure that the mixing length is strictly positive. In this way singularity at the free surface is avoided. The parameter z_0 is related to the roughness of the bed and denotes the zero-intercept level of a log velocity profile for the situation without waves.

The nonlinear system of ODEs describing the (0,0)-problem, has been solved iteratively using a relaxation method, which has been described in Press *et al.* (1992) and is primarily based on ideas, which amongst others can be found in Keller (1968, chapter 3). The linear ODEs for the higher-order problems are solved numerically using the trapezoidal rule, which is of second-order accuracy. Due to the existence of boundary layers near the bottom and the free surface, grid refinement (in the vertical direction) has been carried out in these regions.

6. Application to wave–current channel problems

6.1. *Simplification of present model*

For model verification two test problems are considered, which both concern wave and current motion in a laboratory flume. Since a comparison with measurements in the centre of the flume has been made and influences from the sidewalls are not taken into account, lateral variations (in the horizontal direction perpendicular to the propagation direction, or the y-direction) have been neglected. Furthermore, in all experiments the mass transport through all cross-sectional planes is assumed constant, or even zero if no initial current is generated, in a local area around the measuring point. In a GLM formulation this requirement reads for a rectangular channel

$$\int_{-h}^{\bar{\zeta}^L} \bar{u}^L \mathrm{d}z = Q, \tag{6.1}$$

with Q the prescribed mass transport per unit of channel width. Since the measurements have been carried out in only one cross-section of the flume and we are interested only in the local solution of flow field, information about the free-surface elevation has to be specified. Therefore the mean free-surface elevation has been chosen equal to zero, $\hat{\zeta}^{(1,0)} = \hat{\zeta}^{(2,0)} = 0$ and the amplitude function of the fluctuating part of the free surface is set equal to the measured wave amplitude, $\hat{\zeta}^{(1,1)} = a$.

The zeroth-order solution is supposed to be steady and uniform in the horizontal direction, thus only dependent on the vertical coordinate z. The horizontal momentum equation (5.7a) yields a linear shear stress distribution in the current direction, or x-direction,

$$\hat{\tau}_{13}^{(0,0)} = \hat{\tau}_b^{(0,0)} \left(\frac{-z}{h} \right), \tag{6.2}$$

with $\hat{\tau}_b^{(0,0)}$ the bottom shear stress for the uniform current. The constant $\hat{\tau}_b^{(0,0)}$ is chosen such that for a given mass transport $\hat{Q}^{(0,0)}$, the horizontal zeroth-order velocity $\hat{u}^{(0,0)}$ satisfies

$$\hat{Q}^{(0,0)} = \int_{-h}^{0} \hat{u}^{(0,0)} \mathrm{d}z. \tag{6.3}$$

The hydrostatic pressure distribution is found from the vertical momentum equation,

$$\hat{p}^{(0,0)} = -\rho g z. \tag{6.4}$$

As described at the end of § 5.2 the nonlinear equations (5.17)–(5.19) together with the relations (6.2)–(6.4) are solved numerically. For a more detailed description of this $(0,0)$-problem see Klopman (1992).

Since at first order ($n = 1, -1 \leqslant m \leqslant 1$) all the forcing terms are zero, the solution for the mean motion at first order is completely determined by the mean free-surface elevation $\hat{\zeta}^{(1,0)}$ in the sense that the amplitude variables can be written as

$$\hat{\varphi}^{(1,0)}(\boldsymbol{X}_h, T, z) = \Phi(z)\, \hat{\zeta}^{(1,0)}(\boldsymbol{X}_h, T). \tag{6.5}$$

For an arbitrary value of $\hat{\zeta}^{(1,0)}$ the form function $\Phi(z)$ of each variable $\hat{\varphi}^{(1,0)}$ is determined numerically. Although $\hat{\zeta}^{(1,0)} = 0$ and thus $\hat{\varphi}^{(1,0)} = 0$, their temporal and spatial derivatives in the horizontal direction are not necessarily equal to zero.

The first-order first-harmonic solution represents the carrier wave solution. This problem is solved most easily by introducing related variables

$$\breve{\varphi}^{(1,1)} = \hat{\varphi}^{(1,1)} - \hat{\eta}^{(1,1)} \frac{\partial \hat{\varphi}^{(1,1)}}{\partial z}, \tag{6.6}$$

which up to first order can be seen as the Eulerian counterpart of the amplitude function of the generalized Lagrangian variable, $\hat{\varphi}^{(1,1)}$. For laminar flow ($\nu = \nu_0$) the equations for $\breve{\varphi}^{(1,1)}$ reduce to the so-called Orr–Sommerfeld equation, which is often used in the study of hydrodynamic stability, see e.g. Drazin & Reid (1984). The carrier wave solution will show thin wave boundary layers near the bottom and the free surface. Outside the boundary layers the velocities will be close to the velocities obtained with the linear potential-flow theory. For a given frequency ω, the unknown wavenumber k can be determined and will be complex valued. The imaginary part of k reflects the wave decay due to dissipation.

Our main interest concerns the second-order mean motion ($n = 2, m = 0$). The forcing terms are no longer equal to zero, but contain temporal and (horizontal) spatial derivatives of first-order zeroth-harmonic variables ($n = 1, m = 0$) as well as correlations of wave-related variables ($n = 1, m = \pm 1$). For second- and higher-order problems ($n \geqslant 2$) the homogeneous problem is similar to the zeroth- and first-order problem. Because the forcing terms are non-zero, a constraint must be imposed to avoid secular behaviour of the particular solution. For $m = 0$ this so-called solvability condition reads

$$\int_{-h}^{0} \hat{F}^{(2,0)} \mathrm{d}z = \hat{L}^{(2,0)} - \hat{H}_3^{(2,0)}. \tag{6.7}$$

By substituting (B 5), (B 8), (B 9) into (6.7) and writing $\hat{u}^{(1,0)} = \hat{U}^{(1)}\hat{\zeta}^{(1,0)}$, the solvability condition reduces to a relation between the temporal and horizontal derivatives of the first-order mean surface elevation,

$$\frac{\partial \hat{\zeta}^{(1,0)}}{\partial T} + \hat{u}_\alpha^{(0,0)}\,|_{(z=0)}\,\frac{\partial \hat{\zeta}^{(1,0)}}{\partial X_\alpha} + \left(\int_{-h}^{0} \hat{U}^{(1)} \mathrm{d}z\right)\frac{\partial \hat{\zeta}^{(1,0)}}{\partial X_\alpha} = \mathrm{i}\left(k - \bar{k}\right)\int_{-h}^{0} \tilde{u}^{(2,0)} \mathrm{d}z + \tilde{w}^{(2,0)}\,|_{(z=0)}\,. \tag{6.8}$$

Here $\tilde{u}^{(2,0)}$ and $\tilde{w}^{(2,0)}$ denote the second-order approximation of the Stokes correction, given in Appendix B by relation (B 4). By writing each temporal or spatial gradient of a first-order dependent variable as a product of its form function and the gradient of the first-order mean surface elevation, as in (6.5), and substituting (6.8) to remove the temporal gradients from the expressions for the forcing functions, an extra dependent variable $\partial\hat{\zeta}^{(1,0)}/\partial X_\alpha$ is introduced. Therefore, an additional constraint has to be imposed. As for the situation of currents without waves, the gradient of the mean free-surface elevation is chosen such that relation (6.1) still holds at second order. Because the mean free-surface elevation is assumed to be zero, this results in

$$\int_{-h}^{0} \hat{u}^{(2,0)} \mathrm{d}z = 0. \tag{6.9}$$

After evaluation of the driving force, the linear non-homogeneous system of ODEs is solved numerically.

6.2. *Comparison with observations*

As already remarked two different sets of laboratory wave–current channel measurements are used to verify the present model. Firstly, the mean flow generated by a uniform regular wave train is considered. Both the analytical conduction solution presented by Longuet-Higgins (1953) for the horizontal drift, or mass transport for waves, in a viscous fluid, and experimental observations of the drift velocities in a closed wave channel by Mei *et al.* (1972) are taken as a reference. Although these references are based on pure Lagrangian averaging, i.e. averaging by following a single fluid particle, it is nevertheless legitimate to compare the results with the GLM

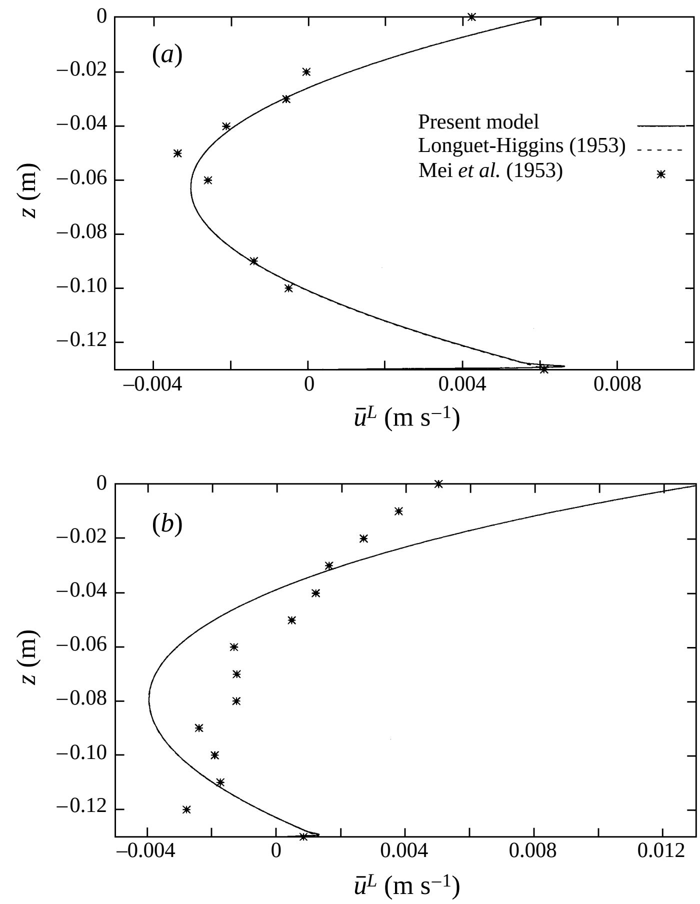

FIGURE 1. Second-order drift velocities: (*a*) $\mathrm{Re}(kh) = 1.02$, (*b*) $\mathrm{Re}(kh) = 1.81$.

results obtained by the present model. This is due to the fact that the present model provides GLM velocities which are of second order, and if there is no initial current, the difference between a pure Lagrangian mean velocity and a GLM velocity can be proven to be of third order.

Note that our model computes the solution over the whole depth at once, whereas Longuet-Higgins used a three-layer approach. He first solved the equations for the mean flow in the boundary layers in order to obtain boundary conditions for the interior problem. Furthermore, Longuet-Higgins (1953) neglected the effect of wave decay. In the present model this effect is taken into account. The wave decay, which is assumed to be spatial and not temporal, is ruled by the imaginary part of the wavenumber k.

Mei *et al.* (1972) generated a regular wave field in a closed 12 m long, 0.76 m wide tank with a still-water depth $h = 13\,\mathrm{cm}$. For waves in a closed channel a constant horizontal pressure gradient is imposed which is chosen to yield zero mean mass flux $Q = 0$. Two sets of measurements from a station 3.5 m from the wavemaker are considered, namely $a = 1.1\,\mathrm{cm}$, $T = 0.81\,\mathrm{s}$ and $a = 0.76\,\mathrm{cm}$, $T = 0.56\,\mathrm{s}$.

For these boundary layer streaming problems the eddy viscosity distribution has been assumed constant over the vertical and chosen equal to the kinematic viscosity, $\nu = 10^{-6}\,\mathrm{m^2\,s^{-1}}$. This means that the second-order GLM velocity should be equal to Longuet-Higgins' conduction solution, if dissipative effects such as wave decay had been neglected. In figure 1 the results from our model are compared with the conduction solution and the observations by Mei *et al.* (1972, figures 4.1*a* and 4.1*l* in their report).

In view of the small difference between the computed solution and the conduction solution, the conclusion is justified that the wave decay has little influence when the flow is viscous and non-turbulent. Craik (1982*a*, p. 201) remarked that significant departures from the conduction solution can be expected when the magnitude of the imaginary part of k becomes comparable with (or greater than) h^{-1}. However, in both test problems this is not the case, since $\mathrm{Im}(kh) = O(10^{-3})$.

Mei *et al.* (1972, p. 152) remarked that for $0.9 \leqslant \mathrm{Re}(kh) \leqslant 1.5$ the measurements agreed with the conduction solution. Figure 1 confirms this statement, since in the two cases $\mathrm{Re}(kh) = 1.02$ and $\mathrm{Re}(kh) = 1.81$ respectively. There might be several reasons for this. As in the analysis of Longuet-Higgins, correlations between mean quantities are neglected in the present model, if no initial current exists. Mei (1989, § 9.5) showed that neglecting nonlinear convective terms might be dangerous, especially if the wave amplitude is of the same magnitude as the boundary layer thickness, which is true for this problem. The present WKBJ expansion, which only takes into account the first harmonic at first order, is only valid for currents that are not weak compared to the wave motion. In fact, if there is no initial current higher harmonics should be taken into account. Since these higher harmonics have been neglected the applied WKBJ method might lead to wrong solutions for boundary layer streaming problems. These higher harmonics might have a greater impact on the solution, as the velocity in the boundary layers at the free surface and the bottom is larger. These velocities are large in magnitude for small and large values of $\mathrm{Re}(kh)$, probably so large that the analysis in the present model does not hold any more.

Secondly, wave-induced changes in an initial turbulent current are considered. Model results are compared with measurements, obtained by Klopman (1994), in a wave–current laboratory flume. Klopman (1994) used laser Doppler velocimetry (LDV) flow meters to measure horizontal and vertical velocities of the total turbulent flow. In the present model a turbulent current with a mean horizontal mass transport velocity of $Q = 0.08\,\mathrm{m\,s^{-1}}$ was generated in a flume with a still-water depth $h = 0.50\,\mathrm{m}$. A monochromatic wave field with a wave period $T = 1.44\,\mathrm{s}$ and wave amplitude $a = 0.060\,\mathrm{m}$ is imposed on the current. The following values for the empirical constants in the turbulent model have been used: $\sigma_k = 1$, $C_D = 0.156$, $C_\mu^{'} = 0.54$. These values are normally used for this type of problems, see Rodi (1984). From the measurements the bed roughness parameter $z_0 = 0.037\,\mathrm{mm}$. The choice $z_a = 1\,\mathrm{mm}$ results in a lengthscale $\ell_0 = 0.01\,\mathrm{m}$ at the still-water level.

In figure 2 the absolute value of the complex-valued amplitude functions of the Eulerian orbital horizontal velocity u' are given for the cases of no current, a following current and an opposing current. The effect of wave decay is taken into account in the model. However, as in the first test problem the wave decay does not play a significant role. In this figure not only are the computed values from the model given, but the values measured by Klopman (1994) as well. The Eulerian and Lagrangian disturbances are related by $u' = u^{\ell} - \xi_k \partial \overline{u}^L / \partial x_k + O(a^2)$. The difference between the two disturbances is at most 1.3%. The model results do not correspond exactly with the measurements, although a qualitative agreement can be observed. The interaction with a following current results in a decrease of the vertical gradient of the amplitude of the horizontal velocity component, while the interaction with an opposing current is shown to increase this vertical gradient.

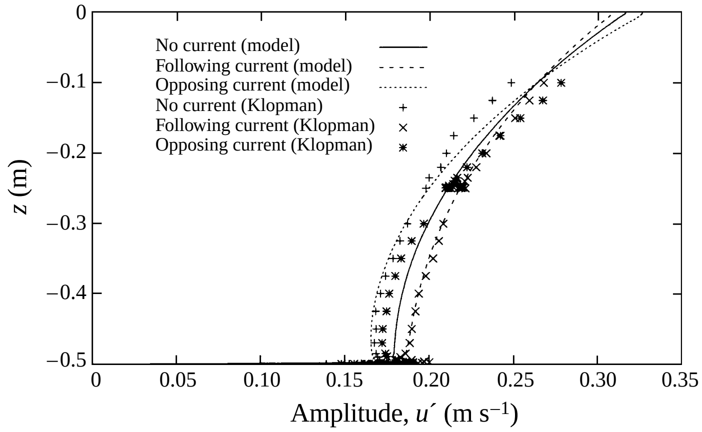

Figure 2. GLM results (present model) and experimental results (Klopman 1994) for the first-order Eulerian horizontal velocity amplitude profile.

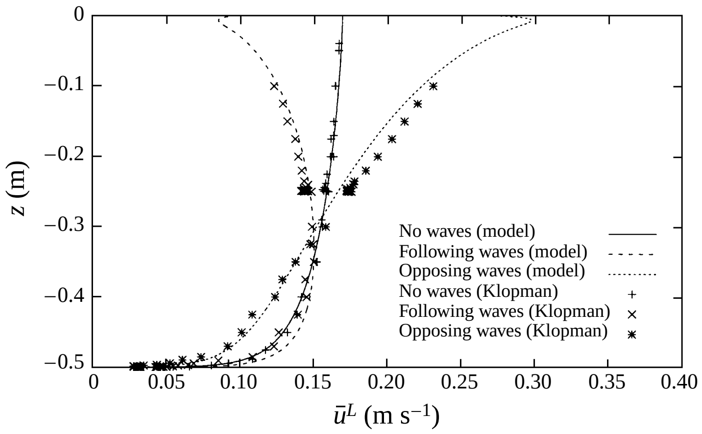

Figure 3. GLM results (present model) and experimental results (Klopman 1994) for the Eulerian-mean horizontal velocity profile.

The modifications of the mean horizontal velocity profile are shown in figure 3. Here the Eulerian-mean velocity profiles for waves following and opposing the current can be compared to the current profile in the situation without waves. In each case the total discharge with waves is the same as without waves. Comparing the model results with the experimental data of Klopman (1994) not only can a qualitative agreement be observed, but the computed velocity profiles show quantitative correspondence as well. The changes of the current velocity profiles due to the presence of following or opposing waves are significant. The waves propagating in the current direction cause a reduction of the mean velocity shear, or vertical gradient of the mean horizontal velocity, whereas waves opposing the current increase the velocity shear. This kind of behaviour has also been reported by Bakker & Van Doorn (1978) and Kemp & Simons (1982, 1983) in their experimental studies on wave–current interaction.

In the boundary layer at the free surface, where no experimental data are available due to the Eulerian measuring procedure, a rather sharp gradient of the horizontal velocity can be observed. This behaviour might be ascribed to the simplified turbulence model, which is not able to describe the wave motion, and as a result the wave-induced driving force for the mean motion, in this boundary layer in a proper way.

7. Conclusions

In this paper the GLM formulation is used to split the oscillating motion from the mean motion over the whole depth in a unique and unambiguous way. GLM equations have been derived for the combined wave–current motion, which differ from the original GLM equations of Andrews & McIntyre (1978*a*). Due to the consideration of viscosity and turbulent motion, the GLM equations derived by Andrews & McIntyre (1978*a*) are difficult to solve exactly; consequently some approximation is necessary. A set of equations which have a similar structure to the equations in Eulerian form has been obtained. As a result, numerical solution techniques solving problems in the conventional Eulerian formulation can be applied. A WKBJ-type perturbation-series approach has been employed to obtain the modification of the amplitude functions of the orbital velocities and the changes of the profiles of the mean velocities, all induced by the wave–current interactions.

Results from the test problems are satisfactory. The wave-induced horizontal drift profiles, which have been obtained for the situation without initial current, agree with Longuet-Higgins' conduction solution. However, these results do not agree with experimental results of Mei *et al.* (1972) in all situations that were considered. Neglect of nonlinear convective terms might be a reason for this.

The changes of the vertical gradient of the horizontal velocity amplitude profile, caused by a following or opposing current, and the changes of the vertical gradient of the mean horizontal velocity, induced by the modified wave field propagating in the current direction or opposite to this direction, match both qualitatively and quantitatively with the experimental data. For these problems it is not useful to go to higher order in the WKBJ expansion. The model verification of the boundary layer streaming problem showed that inclusion of higher harmonics might be necessary. Since these higher harmonics have been neglected in the present WKBJ expansion, the model results will not necessarily be improved by going to higher order of approximation in the GLM equations or higher order of accuracy in the WKBJ expansion. Further study for this type of problems is required. However, this is beyond the scope of the present paper.

In modelling the wave-channel problems, variations in cross-direction have been neglected in the present paper. Since these variations might have some impact on the mean velocities, a three-dimensional description of the motion in GLM formulation is desirable. Several authors, e.g. Dingemans *et al.* (1996) have already shown that in wave–current channels lateral variations might be significant. In order to include effects that are caused by these variations the model has to be extended to three dimensions.

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Appendix A. Alternative derivation of GLM momentum equation

A.1. *Some initial properties for transformation*

In order to deal properly with the transformation from Eulerian to GLM formulation and vice versa via the mapping $\boldsymbol{x} \rightarrow \boldsymbol{x} + \boldsymbol{\xi}$, some important consequences of the chain rule are outlined. First, the chainrule itself yields

$$\frac{\partial \varphi^{\xi}}{\partial x_i} = \frac{\partial \varphi}{\partial \Xi_j}\frac{\partial \Xi_j}{\partial x_i}, \quad \frac{\partial \varphi}{\partial \Xi_i} = \frac{\partial \varphi^{\xi}}{\partial x_j}\frac{\partial x_j}{\partial \Xi_i}. \tag{A 1}$$

The final term $\partial x_j/\partial \Xi_i$ in (A 1) can be expressed in terms of $\partial \Xi_j/\partial x_i \equiv \delta_{ij} + \partial \xi_j/\partial x_i$ by introducing K_{ij} as the cofactors of J, which satisfy

$$\frac{\partial \Xi_i}{\partial x_k}K_{ij} = J\delta_{kj} = \frac{\partial \Xi_k}{\partial x_i}K_{ji}. \tag{A 2}$$

For small wave amplitude a the following approximate expression for the cofactor K_{ij} can be derived from the first relation in (A 2):

$$K_{ij} = J\left(\delta_{ij} - \frac{\partial \xi_j}{\partial x_i} + \frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_j}{\partial x_k}\right) + O(a^3). \tag{A 3}$$

Andrews & McIntyre (1978*a*, p. 640) and, in a more fundamental way Dingemans (1997, p. 240), showed that the second relation in (A 1) can then be rewritten as

$$\frac{\partial \varphi}{\partial \Xi_i} = \frac{1}{J}K_{ij}\frac{\partial \varphi^{\xi}}{\partial x_j}. \tag{A 4}$$

Further properties of J and K_{ij} can be found in Dingemans (1997).

A.2. *Expressions in terms of GLM quantities*

The acceleration term $\mathrm{D}u_i/\mathrm{D}t$ is transformed by using a direct consequence of relation (2.2),

$$\left(\frac{\mathrm{D}\varphi}{\mathrm{D}t}\right)^{\zeta} = \overline{\mathrm{D}}^{L}\left(\varphi^{\xi}\right). \tag{A 5}$$

Since $\mathrm{D}u_i/\mathrm{D}t$ is considered at the fixed point $\boldsymbol{x}$ and relation (A 5) considers the Eulerian material derivative at the disturbed position Ξ, a Taylor series expansion around Ξ is carried out. For some quantity $\varphi(\boldsymbol{x}, t)$ this yields

$$\begin{aligned}\varphi(\boldsymbol{x}, t) &= \varphi(\Xi - \boldsymbol{\xi}, t) \\ &= \varphi(\Xi, t) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\xi_{j_1}\ldots\xi_{j_n}\frac{\partial^n \varphi(\Xi, t)}{\partial \Xi_{j_1}\cdots\partial \Xi_{j_n}}.\end{aligned} \tag{A 6}$$

By applying the chainrule (A 4) to the partial derivatives occurring in the summation in the right-hand side of relation (A 6), the partial derivatives to Ξ_j can be replaced by partial derivatives to x_j. By defining for some quantity $\phi = \phi(\boldsymbol{x}, t)$ the operator

$$\mathscr{T}(\phi, \boldsymbol{\xi}, \boldsymbol{x}, t) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\xi_{j_1}\xi_{j_2}\ldots\xi_{j_n}\frac{1}{J}K_{j_1 m_1}\frac{\partial}{\partial x_{m_1}}\left(\frac{1}{J}K_{j_2 m_2}\frac{\partial}{\partial x_{m_2}}\left[\cdots\frac{1}{J}K_{j_n m_n}\frac{\partial \phi}{\partial x_{m_n}}\right]\right), \tag{A 7}$$

relation (A 6) can be written as

$$\varphi(\boldsymbol{x}, t) = \varphi^{\xi}(\boldsymbol{x}, t) - \mathscr{T}(\varphi^{\xi}, \boldsymbol{\xi}, \boldsymbol{x}, t). \tag{A 8}$$

According to the definition of the Stokes correction, given by (2.7), averaging relation (A 8) shows that an expression in terms of GLM quantities can be obtained by averaging the operator $\mathscr{T}(\varphi^{\xi}, \boldsymbol{\xi}, \boldsymbol{x}, t)$. By substituting (A 3) into (A 8), (A 7) a second-order approximation for the Stokes correction is derived,

$$\overline{\varphi}^S(\boldsymbol{x},t) \equiv \overline{\varphi}^L(\boldsymbol{x},t) - \overline{\varphi}(\boldsymbol{x},t) = \left\langle \xi_j \left(\frac{\partial \varphi^{\xi}}{\partial x_j} - \frac{\partial \xi_k}{\partial x_j}\frac{\partial \varphi^{\xi}}{\partial x_k} \right) \right\rangle - \left\langle \tfrac{1}{2}\xi_j\xi_k \frac{\partial^2 \varphi^{\xi}}{\partial x_j \partial x_k} \right\rangle + O(a^3). \tag{A 9}$$

A.3. *Derivation of GLM equations*

For the alternative derivation of the GLM equations of motion the momentum equation in Eulerian formulation is transformed by applying the definition of the total Stokes correction. The only difficulty concerns the treatment of the acceleration term. However, relation (A 9) can also be exploited to express the $\mathrm{D}u_i/\mathrm{D}t$ entirely in terms of GLM quantities. Substitution of relation (A 5) into the expression that is obtained after expanding the acceleration term around Ξ, yields

$$\frac{\mathrm{D}u_i}{\mathrm{D}t}(\boldsymbol{x},t) = \overline{\mathrm{D}}^L u_i^{\zeta} - \xi_j \frac{\partial}{\partial x_j}\left(\overline{\mathrm{D}}^L u_i^{\zeta}\right) + \xi_j \frac{\partial \xi_k}{\partial x_j}\frac{\partial}{\partial x_k}\left(\overline{\mathrm{D}}^L u_i^{\zeta}\right) + \tfrac{1}{2}\xi_j\xi_k \frac{\partial^2}{\partial x_j \partial x_k}\left(\overline{\mathrm{D}}^L u_i^{\zeta}\right) + O(a^3). \tag{A 10}$$

Extra attention is paid to the second term on the right-hand side of relation (A 10), which upon using (2.6), can be written as

$$\xi_j \frac{\partial}{\partial x_j}\left(\overline{\mathrm{D}}^L u_i^{\zeta}\right) = \frac{\partial}{\partial x_j}\left(\overline{\mathrm{D}}^L \left(\xi_j u_i^{\zeta}\right)\right) - \frac{\partial}{\partial x_j}\left(u_j^{\ell} u_i^{\zeta}\right) - \frac{\partial \xi_j}{\partial x_j}\overline{\mathrm{D}}^L u_i^{\zeta}. \tag{A 11}$$

Substitution of both the acceleration term (A 10) and definition (A 9) of the total Stokes correction into the momentum equation in Eulerian formulation results in a momentum equation in terms of Lagrangian quantities,

$$\begin{aligned}\overline{\mathrm{D}}^L u_i^{\zeta} + \frac{1}{\rho}\frac{\partial p^{\zeta}}{\partial x_i} - \frac{1}{\rho}\frac{\partial \tau_{ij}^{\zeta}}{\partial x_j} - F_i^{\zeta} &= \frac{1}{\rho}\frac{\partial p^S}{\partial x_i} - \frac{1}{\rho}\frac{\partial \tau_{ij}^S}{\partial x_j} - F_i^S + \frac{\partial}{\partial x_j}\left(\overline{\mathrm{D}}^L\left(\xi_j u_i^{\zeta}\right)\right) - \frac{\partial}{\partial x_j}\left(u_j^{\ell} u_i^{\zeta}\right) - \frac{\partial \xi_j}{\partial x_j}\overline{\mathrm{D}}^L u_i^{\zeta} \\ &\quad - \xi_j \frac{\partial \xi_k}{\partial x_j}\frac{\partial}{\partial x_k}\left(\overline{\mathrm{D}}^L u_i^{\zeta}\right) - \tfrac{1}{2}\xi_j\xi_k \frac{\partial^2}{\partial x_j \partial x_k}\left(\overline{\mathrm{D}}^L u_i^{\zeta}\right) + O(a^3).\end{aligned} \tag{A 12}$$

In the right-hand side of this momentum equation the correction φ^S is defined as $\varphi^S = \mathscr{T}(\varphi^{\xi}, \boldsymbol{\xi}, \boldsymbol{x}, t)$. Furthermore, the divergence of the disturbance displacement appears. As long as the Eulerian disturbance velocity field $\boldsymbol{u}' = \boldsymbol{u} - \overline{\boldsymbol{u}}$ is divergence-free, $\partial u_j'/\partial x_j = 0$, the divergence of the disturbance displacement is of second order,

$$\frac{\partial \xi_j}{\partial x_j} = O(a^2). \tag{A 13}$$

For the GLM motion the following equation can be derived by averaging equation

(A 12) and using (A 13):

$$\overline{\mathrm{D}}^L \overline{u}_i^L + \frac{1}{\rho}\frac{\partial \overline{p}^L}{\partial x_i} - \frac{1}{\rho}\frac{\partial \overline{\tau}_{ij}^L}{\partial x_j} - \overline{F}_i^L$$
$$= \frac{1}{\rho}\frac{\partial \overline{p}^S}{\partial x_i} - \frac{1}{\rho}\frac{\partial \overline{\tau}_{ij}^S}{\partial x_j} - \overline{F}_i^S + \frac{\partial}{\partial x_j}\left(\overline{\mathrm{D}}^L\left(\overline{\xi_j u_i^\ell}\right)\right) - \frac{\partial}{\partial x_j}\left(\overline{u_j^\ell u_i^\ell}\right)$$
$$- \left\langle \xi_j \frac{\partial \xi_k}{\partial x_j}\right\rangle \frac{\partial}{\partial x_k}\left(\overline{\mathrm{D}}^L \overline{u}_i^L\right) - \tfrac{1}{2}\overline{\xi_j\xi_k}\frac{\partial^2}{\partial x_j \partial x_k}\left(\overline{\mathrm{D}}^L \overline{u}_i^L\right) + O(a^3). \quad \text{(A 14)}$$

Subtracting equation (A 14) from (A 12) provides the equation for the disturbed Lagrangian motion. Substitution of the fluctuating part of the total Stokes correction into the resulting equation then leads to

$$\overline{\mathrm{D}}^L u_i^\ell + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p^\ell - \xi_j \frac{\partial \overline{p}^L}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\tau_{ij}^\ell - \xi_k \frac{\partial \overline{\tau}_{ij}^L}{\partial x_k}\right) = \xi_j \frac{\partial}{\partial x_j}\left(\overline{\mathrm{D}}^L \overline{u}_i^L\right) + O(a^2). \quad \text{(A 15)}$$

A.4. *Shear stresses*

For the derivation of the GLM shear stresses in terms of GLM velocities, definition (3.1) is used. By definition the shear stresses have to be evaluated at disturbed positions. According to the chain rule (A 4) the following expression for the shear stresses is obtained:

$$\frac{\tau_{ij}^\zeta}{\rho} = \frac{\nu^\zeta}{J}\left(K_{jk}\frac{\partial u_i^\zeta}{\partial x_k} + K_{ik}\frac{\partial u_j^\zeta}{\partial x_k}\right). \quad \text{(A 16)}$$

By averaging (A 16) a second-order approximation of the mean shear stresses is obtained,

$$\frac{\overline{\tau}_{ij}^L}{\rho} = \frac{\overline{\nu}^L}{J}\left(\frac{\partial \overline{u}_i^L}{\partial x_j} - \left\langle \frac{\partial \xi_k}{\partial x_j}\frac{\partial u_i^\ell}{\partial x_k}\right\rangle + \left\langle \frac{\partial \xi_m}{\partial x_j}\frac{\partial \xi_k}{\partial x_m}\right\rangle \frac{\partial \overline{u}_i^L}{\partial x_k}\right.$$
$$\left. + \frac{\partial \overline{u}_j^L}{\partial x_i} - \left\langle \frac{\partial \xi_k}{\partial x_i}\frac{\partial u_j^\ell}{\partial x_k}\right\rangle + \left\langle \frac{\partial \xi_m}{\partial x_i}\frac{\partial \xi_k}{\partial x_m}\right\rangle \frac{\partial \overline{u}_j^L}{\partial x_k}\right)$$
$$+ \frac{1}{J}\left(\left\langle \nu^\ell \frac{\partial u_j^\ell}{\partial x_i}\right\rangle - \left\langle \nu^\ell \frac{\partial \xi_k}{\partial x_i}\right\rangle \frac{\partial \overline{u}_j^L}{\partial x_k} + \left\langle \nu^\ell \frac{\partial u_i^\ell}{\partial x_j}\right\rangle - \left\langle \nu^\ell \frac{\partial \xi_k}{\partial x_j}\right\rangle \frac{\partial \overline{u}_i^L}{\partial x_k}\right) + O(a^3). \quad \text{(A 17)}$$

Subtracting equation (A 17) from (A 16) results in

$$\frac{\tau_{ij}^\ell}{\rho} = \frac{\overline{\nu}^L}{J}\left(\frac{\partial u_i^\ell}{\partial x_j} - \frac{\partial \xi_k}{\partial x_j}\frac{\partial \overline{u}_i^L}{\partial x_k} + \frac{\partial u_j^\ell}{\partial x_i} - \frac{\partial \xi_k}{\partial x_i}\frac{\partial \overline{u}_j^L}{\partial x_k}\right) + \frac{\nu^\ell}{J}\left(\frac{\partial \overline{u}_j^L}{\partial x_i} + \frac{\partial \overline{u}_i^L}{\partial x_j}\right) + O(a^2). \quad \text{(A 18)}$$

The GLM and disturbed eddy viscosity $\overline{\nu}^L$ and ν^ℓ have to be determined by a turbulence model. When the flow is viscous and non-turbulent, $\overline{\nu}^L$ equals the kinematic viscosity and $\nu^\ell = 0$.

Appendix B. Driving forces at different orders of approximation

The explicit solutions will be presented here up to second order. For the zeroth-order problem ($n=0$) the forcing terms are given by

$$\hat{F}^{(0,0)} = \hat{H}_i^{(0,0)} = \hat{K}_i^{(0,0)} = \hat{L}^{(0,0)} = \hat{T}_{ij}^{(0,0)} = 0, \tag{B 1a}$$

$$\hat{G}_\alpha^{(0,0)} = \hat{\tau}_{\alpha b}^{(0,0)}/h, \tag{B 1b}$$

$$\hat{G}_3^{(0,0)} = -g. \tag{B 1c}$$

At first order ($n=1$) all the forcing terms are zero. i.e.

$$\hat{F}^{(1,0)} = \hat{G}_i^{(1,0)} = \hat{H}_i^{(1,0)} = \hat{K}_i^{(1,0)} = \hat{L}^{(1,0)} = \hat{T}_{ij}^{(1,0)} = 0, \tag{B 2}$$

$$\hat{F}^{(1,1)} = \hat{G}_i^{(1,1)} = \hat{D}_i^{(1,1)} = \hat{H}_i^{(1,1)} = \hat{K}_i^{(1,1)} = \hat{L}^{(1,1)} = \hat{T}_{ij}^{(1,1)} = 0. \tag{B 3}$$

The equations for the second-order mean motion ($n=2, m=0$) contain forcing terms which are no longer equal to zero, but contain temporal and (horizontal) spatial derivatives of first-order zeroth-harmonic variables ($n=1, m=0$) as well as correlations of wave-related variables ($n=1, m=\pm 1$). The second-order approximation of the Stokes correction of the variable φ will be denoted as $\tilde{\varphi}^{(2,0)}$. From relation (A 9) the following expression for $\tilde{\varphi}^{(2,0)}$ can be derived:

$$\begin{aligned}\tilde{\varphi}^{(2,0)} = \Bigg[& \mathrm{i}k_\beta \hat{\xi}_\beta^{(1,-1)} \left(\hat{\varphi}^{(1,1)} - \hat{\eta}^{(1,1)} \frac{\partial \hat{\varphi}^{(0,0)}}{\partial z} \right) - \mathrm{i}\bar{k}_\beta \hat{\xi}_\beta^{(1,1)} \left(\hat{\varphi}^{(1,-1)} - \hat{\eta}^{(1,-1)} \frac{\partial \hat{\varphi}^{(0,0)}}{\partial z} \right) \\ & + \hat{\eta}^{(1,-1)} \frac{\partial}{\partial z} \left(\hat{\varphi}^{(1,1)} - \hat{\eta}^{(1,1)} \frac{\partial \hat{\varphi}^{(0,0)}}{\partial z} \right) + \hat{\eta}^{(1,1)} \frac{\partial}{\partial z} \left(\hat{\varphi}^{(1,-1)} - \hat{\eta}^{(1,-1)} \frac{\partial \hat{\varphi}^{(0,0)}}{\partial z} \right) \\ & + \hat{\eta}^{(1,1)} \hat{\eta}^{(1,-1)} \frac{\partial^2 \hat{\varphi}^{(0,0)}}{\partial z^2} \Bigg] E\bar{E},\end{aligned} \tag{B 4}$$

with $\bar{k}_\beta$ the complex conjugate of k_β and $E = \exp(k_\beta x_\beta - \omega t)$. The forcing functions for $n=2, m=0$ are given by

$$\hat{F}^{(2,0)} = -\frac{\partial \hat{u}_\beta^{(1,0)}}{\partial X_\beta} + \mathrm{i}\left(k_\beta - \bar{k}_\beta\right) \tilde{u}_\beta^{(2,0)} + \frac{\partial \tilde{w}^{(2,0)}}{\partial z}, \tag{B 5}$$

$$\begin{aligned}\hat{G}_\alpha^{(2,0)} = & -\frac{\partial \hat{u}_\alpha^{(1,0)}}{\partial T} - \hat{u}_\beta^{(0,0)} \frac{\partial \hat{u}_\alpha^{(1,0)}}{\partial X_\beta} - \frac{1}{\rho} \frac{\partial \hat{p}^{(1,0)}}{\partial X_\alpha} + \frac{1}{\rho} \frac{\partial \hat{\tau}_{\alpha\beta}^{(1,0)}}{\partial X_\beta} \\ & + \mathrm{i}\left(k_\alpha - \bar{k}_\alpha\right) \frac{1}{\rho} \tilde{p}^{(2,0)} - \mathrm{i}\left(k_\beta - \bar{k}_\beta\right) \frac{1}{\rho} \tilde{\tau}_{\alpha\beta}^{(2,0)} - \frac{1}{\rho} \frac{\partial \tilde{\tau}_{\alpha 3}^{(2,0)}}{\partial z} \\ & + E\bar{E} \left[(\omega_0 - \bar{\omega}_0)\left(k_\beta - \bar{k}_\beta\right) \left(\hat{u}_\alpha^{(1,1)} \hat{\xi}_\beta^{(1,-1)} + \hat{u}_\alpha^{(1,-1)} \hat{\xi}_\beta^{(1,1)} \right) \right] \\ & - E\bar{E} \left[\mathrm{i}(\omega_0 - \bar{\omega}_0) \frac{\partial}{\partial z} \left(\hat{u}_\alpha^{(1,1)} \hat{\eta}^{(1,-1)} + \hat{u}_\alpha^{(1,-1)} \hat{\eta}^{(1,1)} \right) \right] \\ & + E\bar{E} \left[\mathrm{i}\left(k_\beta - \bar{k}_\beta\right) \frac{\partial \hat{u}_\beta^{(0,0)}}{\partial z} \left(\hat{u}_\alpha^{(1,1)} \hat{\eta}^{(1,-1)} + \hat{u}_\alpha^{(1,-1)} \hat{\eta}^{(1,1)} \right) \right] \\ & - E\bar{E} \left[\mathrm{i}\left(k_\beta - \bar{k}_\beta\right) \left(\hat{u}_\alpha^{(1,1)} \hat{u}_\beta^{(1,-1)} + \hat{u}_\alpha^{(1,-1)} \hat{u}_\beta^{(1,1)} \right) \right] \\ & - E\bar{E} \left[\frac{\partial}{\partial z} \left(\hat{u}_\alpha^{(1,1)} \hat{w}^{(1,-1)} + \hat{u}_\alpha^{(1,-1)} \hat{w}^{(1,1)} \right) \right],\end{aligned} \tag{B 6}$$

$$
\begin{aligned}
\hat{G}_3^{(2,0)} = & -\frac{\partial \hat{w}^{(1,0)}}{\partial T} - \hat{u}_\beta^{(0,0)} \frac{\partial \hat{w}^{(1,0)}}{\partial X_\beta} - \frac{1}{\rho} \frac{\partial \hat{\tau}_{3\beta}^{(1,0)}}{\partial X_\beta} \\
& + \frac{1}{\rho} \frac{\partial \tilde{p}^{(2,0)}}{\partial z} - \mathrm{i} \left(k_\beta - \bar{k}_\beta\right) \frac{1}{\rho} \tilde{\tau}_{3\beta}^{(2,0)} - \frac{1}{\rho} \frac{\partial \tilde{\tau}_{33}^{(2,0)}}{\partial z} \\
& + E\bar{E} \left[(\omega_0 - \bar{\omega}_0) \left(k_\beta - \bar{k}_\beta\right) \left(\hat{w}^{(1,1)} \hat{\xi}_\beta^{(1,-1)} + \hat{w}^{(1,-1)} \hat{\xi}_\beta^{(1,1)} \right) \right] \\
& - E\bar{E} \left[\mathrm{i} (\omega_0 - \bar{\omega}_0) \frac{\partial}{\partial z} \left(\hat{w}^{(1,1)} \hat{\eta}^{(1,-1)} + \hat{w}^{(1,-1)} \hat{\eta}^{(1,1)} \right) \right] \\
& + E\bar{E} \left[\mathrm{i} \left(k_\beta - \bar{k}_\beta\right) \frac{\partial \hat{u}_\beta^{(0,0)}}{\partial z} \left(\hat{w}^{(1,1)} \hat{\eta}^{(1,-1)} + \hat{w}^{(1,-1)} \hat{\eta}^{(1,1)} \right) \right] \\
& - E\bar{E} \left[\mathrm{i} \left(k_\beta - \bar{k}_\beta\right) \left(\hat{w}^{(1,1)} \hat{u}_\beta^{(1,-1)} + \hat{w}^{(1,-1)} \hat{u}_\beta^{(1,1)} \right) \right] \\
& - 2\, E\bar{E} \left[\frac{\partial}{\partial z} \left(\hat{w}^{(1,1)} \hat{w}^{(1,-1)} \right) \right],
\end{aligned} \tag{B 7}
$$

$$
\hat{H}_i^{(2,0)} = 0, \tag{B 8}
$$

$$
\hat{L}^{(2,0)} = -\frac{\partial \hat{\zeta}^{(1,0)}}{\partial T} - \hat{u}_\beta^{(0,0)} |_{(z=0)} \frac{\partial \hat{\zeta}^{(1,0)}}{\partial X_\beta}, \tag{B 9}
$$

$$
\begin{aligned}
\hat{K}_\alpha^{(2,0)} = & -\hat{\zeta}^{(1,0)} \frac{\partial \hat{\tau}_{\alpha 3}^{(1,0)}}{\partial z} - E\bar{E} \left[\mathrm{i} k_\alpha \, \hat{\zeta}^{(1,1)} \, \hat{p}^{(1,-1)} - \mathrm{i} \bar{k}_\alpha \, \hat{\zeta}^{(-1,1)} \, \hat{p}^{(1,1)} \right] \\
& + E\bar{E} \left[\mathrm{i} k_\beta \, \hat{\zeta}^{(1,1)} \, \hat{\tau}_{\alpha\beta}^{(1,-1)} - \mathrm{i} \bar{k}_\beta \, \hat{\zeta}^{(-1,1)} \, \hat{\tau}_{\alpha\beta}^{(1,1)} \right],
\end{aligned} \tag{B 10}
$$

$$
\hat{K}_3^{(2,0)} = -\hat{\zeta}^{(1,0)} \left(-\frac{\partial \hat{p}^{(1,0)}}{\partial z} + \frac{\partial \hat{\tau}_{33}^{(1,0)}}{\partial z} \right) + E\bar{E} \left[\mathrm{i} k_\beta \, \hat{\zeta}^{(1,1)} \, \hat{\tau}_{3\beta}^{(1,-1)} - \mathrm{i} \bar{k}_\beta \, \hat{\zeta}^{(-1,1)} \, \hat{\tau}_{3\beta}^{(1,1)} \right], \tag{B 11}
$$

$$
\begin{aligned}
\hat{T}_{\alpha\beta}^{(2,0)} = & \ \hat{v}^{(0,0)} \left(\frac{\partial \hat{u}_\alpha^{(1,0)}}{\partial X_\beta} + \frac{\partial \hat{u}_\beta^{(1,0)}}{\partial X_\alpha} \right) - \hat{v}^{(0,0)} E\bar{E} \left[\bar{k}_\beta \, k_\gamma \, \hat{\xi}_\gamma^{(1,-1)} \, \hat{u}_\alpha^{(1,1)} + k_\beta \bar{k}_\gamma \, \hat{\xi}_\gamma^{(1,1)} \, \hat{u}_\alpha^{(1,-1)} \right. \\
& \qquad \left. - \mathrm{i} \bar{k}_\beta \, \hat{\eta}^{(1,-1)} \, \frac{\partial \hat{u}_\alpha^{(1,1)}}{\partial z} + \mathrm{i} k_\beta \, \hat{\eta}^{(1,1)} \, \frac{\partial \hat{u}_\alpha^{(1,-1)}}{\partial z} \right] \\
& - \hat{v}^{(0,0)} E\bar{E} \left[\bar{k}_\alpha k_\gamma \, \hat{\xi}_\gamma^{(1,-1)} \, \hat{u}_\beta^{(1,1)} + k_\alpha \bar{k}_\gamma \, \hat{\xi}_\gamma^{(1,1)} \, \hat{u}_\beta^{(1,-1)} \right. \\
& \qquad \left. - \mathrm{i} \bar{k}_\alpha \, \hat{\eta}^{(1,-1)} \, \frac{\partial \hat{u}_\beta^{(1,1)}}{\partial z} + \mathrm{i} k_\alpha \, \hat{\eta}^{(1,1)} \, \frac{\partial \hat{u}_\beta^{(1,-1)}}{\partial z} \right] \\
& + \hat{v}^{(0,0)} E\bar{E} \left[\bar{k}_\beta k_\gamma \, \hat{\xi}_\gamma^{(1,-1)} \, \hat{\eta}^{(1,1)} + k_\beta \bar{k}_\gamma \, \hat{\xi}_\gamma^{(1,1)} \, \hat{\eta}^{(1,-1)} \right. \\
& \qquad \left. - \mathrm{i} \bar{k}_\beta \, \hat{\eta}^{(1,-1)} \, \frac{\partial \hat{\eta}^{(1,1)}}{\partial z} + \mathrm{i} k_\beta \, \hat{\eta}^{(1,1)} \, \frac{\partial \hat{\eta}^{(1,-1)}}{\partial z} \right] \frac{\partial \hat{u}_\alpha^{(0,0)}}{\partial z} \\
& + \hat{v}^{(0,0)} E\bar{E} \left[\bar{k}_\alpha k_\gamma \, \hat{\xi}_\gamma^{(1,-1)} \, \hat{\eta}^{(1,1)} + k_\alpha \bar{k}_\gamma \, \hat{\xi}_\gamma^{(1,1)} \, \hat{\eta}^{(1,-1)} \right. \\
& \qquad \left. - \mathrm{i} \bar{k}_\alpha \, \hat{\eta}^{(1,-1)} \, \frac{\partial \hat{\eta}^{(1,1)}}{\partial z} + \mathrm{i} k_\alpha \, \hat{\eta}^{(1,1)} \, \frac{\partial \hat{\eta}^{(1,-1)}}{\partial z} \right] \frac{\partial \hat{u}_\beta^{(0,0)}}{\partial z},
\end{aligned} \tag{B 12}
$$

$$\hat{T}^{(2,0)}_{\alpha 3} = \hat{\nu}^{(0,0)} \frac{\partial \hat{w}^{(1,0)}}{\partial X_\alpha} - \hat{\nu}^{(0,0)} E\bar{E} \left[\mathrm{i}k_\gamma \frac{\partial \hat{\xi}^{(1,-1)}_\gamma}{\partial z} \hat{u}^{(1,1)}_\alpha - \mathrm{i}\bar{k}_\gamma \frac{\partial \hat{\xi}^{(1,1)}_\gamma}{\partial z} \hat{u}^{(1,-1)}_\alpha \right.$$
$$\left. + \frac{\partial \hat{\eta}^{(1,-1)}}{\partial z} \frac{\partial \hat{u}^{(1,1)}_\alpha}{\partial z} + \frac{\partial \hat{\eta}^{(1,1)}}{\partial z} \frac{\partial \hat{u}^{(1,-1)}_\alpha}{\partial z} \right]$$
$$- \hat{\nu}^{(0,0)} E\bar{E} \left[\bar{k}_\alpha k_\gamma \hat{\xi}^{(1,-1)}_\gamma \hat{w}^{(1,1)} + k_\alpha \bar{k}_\gamma \hat{\xi}^{(1,1)}_\gamma \hat{w}^{(1,-1)} \right.$$
$$\left. - \mathrm{i}\bar{k}_\alpha \hat{\eta}^{(1,-1)} \frac{\partial \hat{w}^{(1,1)}}{\partial z} + \mathrm{i}k_\alpha \hat{\eta}^{(1,1)} \frac{\partial \hat{w}^{(1,-1)}}{\partial z} \right]$$
$$+ \hat{\nu}^{(0,0)} E\bar{E} \left[\mathrm{i}k_\gamma \frac{\partial \hat{\xi}^{(1,-1)}_\gamma}{\partial z} \hat{\eta}^{(1,1)} - \mathrm{i}\bar{k}_\gamma \frac{\partial \hat{\xi}^{(1,1)}_\gamma}{\partial z} \hat{\eta}^{(1,-1)} + 2 \frac{\partial \hat{\eta}^{(1,1)}}{\partial z} \frac{\partial \hat{\eta}^{(1,-1)}}{\partial z} \right] \frac{\partial \hat{u}^{(0,0)}_\alpha}{\partial z}, \quad \text{(B 13)}$$

$$\hat{T}^{(2,0)}_{33} = -2\, \hat{\nu}^{(0,0)} E\bar{E} \left[\mathrm{i}k_\gamma \frac{\partial \hat{\xi}^{(1,-1)}_\gamma}{\partial z} \hat{w}^{(1,1)} - \mathrm{i}\bar{k}_\gamma \frac{\partial \hat{\xi}^{(1,1)}_\gamma}{\partial z} \hat{w}^{(1,-1)} \right.$$
$$\left. + \frac{\partial \hat{\eta}^{(1,-1)}}{\partial z} \frac{\partial \hat{w}^{(1,1)}}{\partial z} + \frac{\partial \hat{\eta}^{(1,1)}}{\partial z} \frac{\partial \hat{w}^{(1,-1)}}{\partial z} \right]. \quad \text{(B 14)}$$

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